Bias Reduction for Worst Case based Stress Tests using EVT

Jon Danielsson * Lerby M. Ergun† Casper G. de Vries ‡

November 2016

Abstract

Given that the risk distribution has a heavy tail, the non-parametric worst case analysis, i.e the minimum of the sample, is always downwards biased. Relying on semi-parametric EVT reduces the bias considerably in the case of very heavy tails. For the less heavy tails this relationship is reversed. We derive the bias for the non-parametric heavy tailed order statistics and contrast it with the semi-parametric EVT approach. Estimates for a large sample of US stock returns indicates that this patterns in the bias are also present in financial data. With respect to risk management, this induces an overly conservative capital allocation.

*London School of Economics and Political Science, Systemic Risk Centre.

†Tinbergen Institute, Erasmus University Rotterdam, London School of Economics and Political Science, Systemic Risk Centre and NWO Mozaiek Grant.

‡Tinbergen Institute and Erasmus University Rotterdam

1 Introduction

Many applications in economics and finance depend implicitly or explicitly on worst case analysis. Two examples are the Expected Shortfall (ES) measure in the Basel III market risk proposal and the stress test stipulated by the European Insurance and Occupational Pensions Authority (EIOPA). Another application is the recent global bank stress tests that are based on exogenously imposed large shocks to economic variables. These subjective extreme shocks are often derived from the worst cases that have historically been observed. The maintained assumption in these applications is that a simple statistic, like a discretely observed data point such as the worst realization in a sample, is an unbiased estimator of the underlying theoretical quantile. We evaluate the accuracy of worst case analysis under a variety of realistic distributional assumptions. We ascertain when worst case analysis is reliable and what the biases are.

In this paper we address the following issues concerning the worst case. First, we derive the bias of the non-parametric quantile estimator under the extreme value conditions and compare it to the bias of the semi-parametric quantile estimator. This is done for the class of heavy tailed distributions for which the tail of the distribution is asymptotic to a power law $x^{-\alpha}$.[1] We show that for heavy tailed distributions the method with the smallest bias depends on the tail parameters of the distribution. To support the theoretical analysis we employ Monte Carlo simulations with specific heavy tailed distributions. Subsequently, we use individual stock return data to demonstrate the implications for practical applications.

The non-parametric quantile estimator is the sample extreme order statistic. We apply Extreme Value Theory (EVT) to determine the distribution of the largest order statistic as an estimator of the worst case for heavy tailed distributions. In case of the semi-parametric quantile estimator we follow Hill's (1975) approach to estimate the Value-at-Risk (VaR) at $1/n$; Goldie and Smith (1987) provide the asymptotic mean and variance. Via a Poisson approximation Leadbetter, Lindgren, and Rootzén (1983) derive the asymptotic distribution for the lower order statistics, which we utilize as well.

The bias of the semi-parametric and non-parametric quantile estimator is derived under the assumption that the second order term in the asymptotic expansion of the tail of the distribution also follows a power law. Many known

[1] As in the case of the Pareto distribution and where x denote the losses.

heavy tailed distributions, like the Student-t and Symmetric Stable distributions, satisfy this expansion. Asymptotically the semi-parametric quantile estimator gives a smaller bias than the non-parametric worst case estimate. In finite samples for larger α's (thinner tails) this ranking is reversed.[2]

Danielsson (2011) gives a comprehensive overview of the different methodologies and issues regarding VaR estimation. However, on the specific case of the worst case analysis, the economic literature to date is rather sparse. Nevertheless, worst case analysis is gaining in popularity in applications. An example is the EIOPA (2014) stress test. One of the scenarios in the stress test requires insurers to evaluate a one-in-hundred year adverse shock coming from both the EU equity and corporate bond market.

There are a few indirect examples where the maximum of risk measures are put to use. The Basel committee recently adopted a new framework for the estimation of risk. In their new approach they moved from VaR as a risk measure to ES.[3] In the revised market risk framework they use a function of ES to determine the trading book capital requirements. They use the stressed ES scaled by a ratio of the 12 month ES. The stressed ES is taken as the maximum ES measured over a longer time series.[4] Taking the maximum over a long time horizon calibrates the measure to past crisis periods instead of current market conditions.

The academic literature also focuses on maximum risk measures. Instead of using ES, Ghaoui, Oks, and Oustry (2003) use the maximum VaR over a random space of probability distributions for robust portfolio optimization. Zhu and Fukushima (2009) extend their paper by including ES as the basis of the risk measure. Along this line, Kerkhof, Melenberg, and Schumacher (2010) use the worst case across classes of models to incorporate model risk in capital reserve requirements.

To investigate the implications of the bias for the two different estimators we use the securities database by the Center for Research in Security Price (CRSP) to do worst case analysis. We estimate the worst case with the two quantile estimators for a fixed sample size. Subsequently, we compare the two quantile estimates. We evaluate to what extent the difference between the estimates changes as the empirical distribution becomes less heavily tailed.

[2]For any α there is a sample size $n(\alpha)$ such that for $n > n(\alpha)$ the semi-parametric estimate implies a smaller bias.

[3]As stated in their consultative report. See BIS (2013).

[4]This is the maximum over a rolling window.

We find that the non-parametric quantile estimate is smaller than the semi-parametric quantile estimate for the light heavy tailed stocks. This relationship is reversed for the stocks with a heavier tail. This is an indication that the predicted dynamics in the relative bias of the quantile estimates can also be found in financial return data.

The analysis starts with introducing the two quantile estimators. We derive the biases and show the evolution of the relative biases as a function of the tail parameters. In the subsequent section we explore the extent of the bias via Monte Carlo simulations and use CRSP securities data to find this effect in real world data. In the last section we conclude.

2 Quantile estimators

In this section the two different quantile estimators are introduced. Using EVT the distribution of the largest order statistics, i.e the non-parametric quantile estimator, is derived. This is followed by the introduction of the EVT based semi-parametric quantile estimator.

2.1 Empirical distribution

The limit distribution of the normalized maximum in large samples is provided by the fundamental theorem of EVT. Furthermore, the distribution of the intermediate order statistics follow from the Poisson approximation of the limit law of the number of exceedances as in Leadbetter et al. (1983).

2.1.1 Distribution of the maximum

Given an ordered sample from an i.i.d. non-degenerate cdf F the distribution of the order statistics reads

$$\mathrm{F}_{k,n}(x) = \sum_{r=0}^{k-1} \binom{n}{r} (1-\mathrm{F}(x))^r \, \mathrm{F}^{n-r}(x), \tag{1}$$

where n is the sample size and $\mathrm{F}_{k,n}$ is the cdf of the k^{th} order statistic. For the cdf of the maximum,

$$\mathrm{F}_{n,n}(x) = \mathrm{P}\left(\max\left(X_1, ..., X_n\right) < x\right) = \prod\nolimits_{i=1}^{n} \mathrm{P}\left\{X_i \leq x\right\} = \left[\mathrm{F}(x)\right]^n.$$

EVT gives the conditions under which there exist sequences b_n and a_n such that

$$\lim_{n \to \infty} \left[\mathrm{F}\left(a_n x + b_n \right) \right]^n \to \mathrm{G}(x),$$

where $\mathrm{G}(x)$ is a well defined non-degenerate cdf.[5]

In addition, EVT provides the functional form of the cdf $\mathrm{G}(x)$ that occurs as the limit cdf (Fisher and Tippett (1928) and Gnedenko (1943)). There are three possible $\mathrm{G}(x)$, depending on the shape of the tail of $\mathrm{F}(x)$. In this paper we focus on the Fréchet distribution. The Fréchet limit distribution reads,

$$\text{Fréchet: } \mathrm{G}(x) = \begin{cases} 0, & x \leq 0 \\ \exp\left\{-x^{-\alpha}\right\}, & x > 0 \end{cases} \quad \alpha > 0.$$

We are interested in the cdf of the maxima, $\mathrm{G}\left({}^{(x-b_n)}/_{a_n} \right)$. The parameter α is the shape parameter and is referred to as the tail index. The cdf of the maximum is used to derive the expectation and the median of the maximum. We utilize these two statistics of centrality to make statements about the behavior of the quantile estimators.

Let $X = \max(X_1, X_2, ..., X_n)$, where the X_i are Fréchet distributed. Then the expectation and the median of the maximum for the Fréchet distribution are,[6]

$$\text{Fréchet:} \begin{cases} \mathrm{E}[X] = b_n + a_n \Gamma\left(1 - \frac{1}{\alpha}\right) \\ \text{Median}[X] = b_n + a_n \log(2)^{-1/\alpha} \end{cases}$$

The scaling constants a_n and b_n take different forms for specific parametric distributions within the domain of attraction of the Fréchet distribution. Some examples of scaling constants for heavy tailed parametric distributions are presented in Table 1. The Student-t, Pareto and Symmetric Stable distribution are used in our Monte Carlo simulations.

2.1.2 Distribution of lower order statistics

The cdf of the intermediate order statistics of an i.i.d. sample are represented by (1). Finding the expectation of the order statistic by assuming a particular parametric distribution is not a trivial matter. Leadbetter et al. (1983)[7]

[5] One says that $\mathrm{F}(\cdot)$ falls in the domain of attraction of $\mathrm{G}(\cdot)$.

[6] The derivations are in Appendix A.1.

[7] See page 33 Theorem 2.2.2 in Leadbetter et al. (1983).

Table 1: EVT norming constants for $\alpha > 0$

Distribution family	a_n	b_n
Pareto	$n^{1/\alpha}$	0
Stable	$\left[n\frac{1}{\pi}\sin\left(\frac{\alpha\pi}{2}\right)\Gamma\left(\alpha\right)\right]^{1/\alpha}$	0
Student-t	$\left[n\frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\sqrt{\alpha\pi}}\alpha^{\left(\frac{\alpha-1}{2}\right)}\right]^{1/\alpha}$	0

extend the EVT for the maximum to the lower order statistic by means of the Poisson property of the lower order statistics. The asymptotic distribution of the k^{th} largest order statistic follows as

$$\mathrm{P}\left(X_{n-k+1,n} \leq x\right) \rightarrow \mathrm{G}\left(x\right)\sum_{s=o}^{k-1}\frac{\left(-\log\left[\mathrm{G}\left(x\right)\right]\right)^{s}}{s!},$$

where $X_{n-k+1,n}$ is the k^{th} order statistic. Given this Poisson approximation the cdf for the lower order statistics can be derived.

For a sequence of heavy tailed random variables we have that,

$$\mathrm{G}\left(x\right)\sum_{s=o}^{k-1}\frac{\left(-\log\left[\mathrm{G}\left(x\right)\right]\right)^{s}}{s!} \approx e^{-a_n^{\alpha}x^{-\alpha}}\sum_{s=0}^{k-1}\frac{\left(a_n^{\alpha}x^{-\alpha}\right)^{s}}{s!}.$$

For the density we find

$$g\left(x\right) \approx \alpha a_n^{\alpha}x^{-\alpha-1}e^{-a_n^{\alpha}x^{-\alpha}}\left[\frac{\left(a_n^{\alpha}x^{-\alpha}\right)^{k-1}}{[k-1]!}\right].$$

Given the density of $X_{n-k+1,n}$, we can determine the expectation

$$\mathrm{E}\left[X_{n-k+1,n}\right] = \int_{0}^{\infty} x\alpha a_n^{\alpha}x^{-\alpha-1}e^{-a_n^{\alpha}x^{-\alpha}}\left[\frac{\left(a_n^{\alpha}x^{-\alpha}\right)^{k-1}}{[k-1]!}\right]dx.$$

Applying a change of variable $y = a_n^{\alpha}x^{-\alpha}$, so that $dy = -\alpha a_n^{\alpha}x^{-\alpha-1}$, and a switch of limits we arrive at the following asymptotic approximation

$$\begin{aligned} \mathrm{E}\left[X_{n-k+1,n}\right] &\approx \frac{a_n}{k-1}\int_0^\infty y^{\frac{1}{\alpha}} y^{k-1} e^{-y} dy \\ &= \frac{a_n}{[k-1]!}\Gamma\left[k-\frac{1}{\alpha}\right]. \end{aligned}$$

For the Pareto distribution, i.e. $\mathrm{H}(-x) = Ax^{-\alpha}$, the scaling constant a_n is $(An)^{\frac{1}{\alpha}}$, where A is the scale parameter. The Pareto distribution coincides with the first order term of the so called Hall expansion, i.e

$$\mathrm{P}\left(X \leq -x\right) = \mathrm{F}(-x) = Ax^{-\alpha}[1 + Bx^{-\beta} + o(x^{-\beta})]. \tag{2}$$

Many of the heavy tailed distributions satisfy the Hall expansion. These are therefore approximated by the first order term of the Hall expansion. This allows us to determine the expectation for a wide range of distributions within the domain of attraction of the Fréchet distribution.

2.2 Semi-parametric EVT quantiles.

From the first order expansion in (2) one obtains the approximations to the tail quantiles

$$\mathrm{P}\left(X \geq x\right) = Ax^{-\alpha} \rightarrow \left(\frac{\mathrm{P}\left(X \geq x\right)}{A}\right)^{\frac{1}{-\alpha}} = x_p. \tag{3}$$

In order to estimate x_p in (3) consider a quantile y which is somewhat closer to the center, but sufficiently deep into the tail area such that

$$\left(\frac{\mathrm{P}\left(X \geq y\right)}{A}\right)^{\frac{1}{-\alpha}} \approx y \tag{4}$$

is still a good approximation. Dividing (3) by (4) gives

$$\frac{x}{y} = \left(\frac{\mathrm{P}\left(X \geq x\right)}{\mathrm{P}\left(Y \geq y\right)}\right)^{1/\alpha}. \tag{5}$$

Subsequently, replace the ratio of the two probabilities by their empirical counterparts. Bring y to the other side and replace it by the k^{th} order statistic. This results in the following quantile estimator

$$\hat{x}\left(j,k\right) = x_{n-k+1,n}\left(\frac{k}{j}\right)^{\frac{1}{\hat{\alpha}_k}},$$

where k is the number of order statistics beyond the $x_{n-k-1,n}$ threshold in the tail of the distribution. Here j is the j^{th} order statistic $X_1 \leq X_2 \leq ... \leq X_j \leq ... \leq X_n$ such that $^j/_n$ comes closest to the probability level $\mathrm{P}\left(X < -x\right)$.

Goldie and Smith (1987) derive of the properties of the Hill estimator the distribution of the semi-parametric quantile estimator,

$$\frac{\sqrt{k}}{\log\left(k/np\right)}\left(\frac{\widehat{x}_p}{x_p}-1\right) \sim N\left(-\frac{sign\left(B\right)}{\sqrt{2\beta\alpha}}, \frac{1}{\alpha^2}\right).$$

Here B and β are the second order scale and shape parameter, respectively, from expansion (2).

2.3 Comparing the empirical- and semi-parametric quantile estimator

The asymptotic bias of the semi-parametric quantile estimator at $^1/_n$ is given by,

$$\left(\frac{\widehat{x}_p}{x_p}-1\right) \sim -\frac{sign\left(B\right)}{\sqrt{2\beta\alpha}}\frac{\log\left(k\right)}{\sqrt{k}}. \tag{6}$$

The asymptotic bias of the non-parametric quantile estimator is,[8]

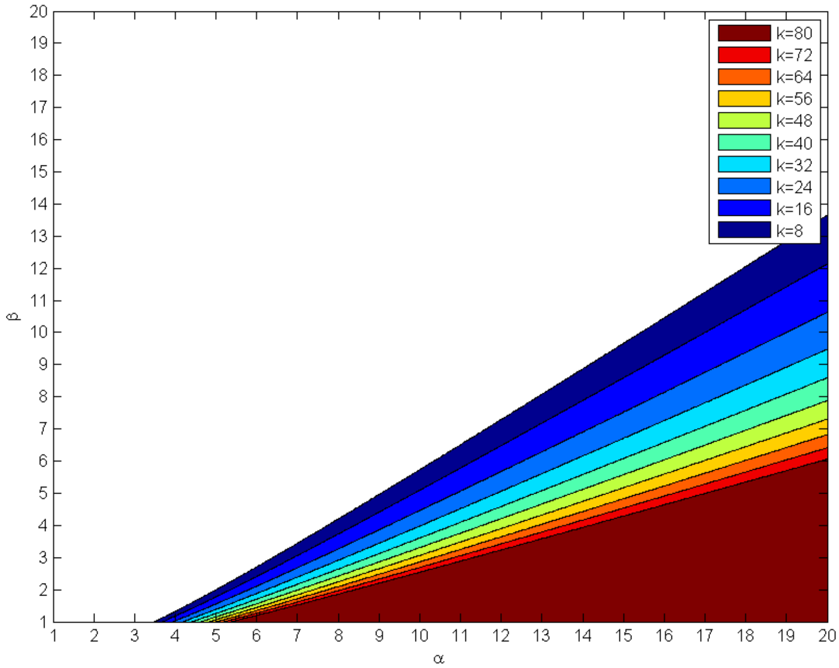
$$\left(\frac{\widetilde{x}_p}{x_p}-1\right) \sim \Gamma\left(1-\frac{1}{\alpha}\right)-1. \tag{7}$$

Notice that for the non-parametric quantile estimator the asymptotic bias approaches infinity as α approaches 1. However, as α increases the Γ function decreases rapidly. The absolute value of the bias of the semi-parametric quantile estimator is relatively small for reasonable values of β and k.[9] For given values of k, β, and $\alpha < \alpha^*$, the absolute bias of the semi-parametric estimator is smaller than the non-parametric quantile estimator. When $\alpha > \alpha^*$, the relationship is reversed. This indicates that for more heavy tailed distributions the absolute bias of the semi-parametric estimator is smaller. This relationship is further depicted in Figure 1. Given various levels of k, Figure 1 depicts at which combination of α and β the bias of the order statistics becomes smaller than the semi-parametric approach.

[8] For the derivation see Appendix A.1.

[9] In applications with daily financial data, k is often smaller than 1,000. For the Student-t distribution $\beta = 2$, but for the Fréchet distribution $\beta = \alpha$. Estimates from financial return data for α are often between 2 and 5.

Figure 1: Bias comparison



This Figure depicts the area where the absolute bias of the semi-parametric estimator becomes larger than the bias of the order statistics (colored region). The biases of the estimators are at $p = 1/n$ as in (6) and (7). For this figure we fix the first k at 8 and then proceed by steps of 8 to $k = 80$. This creates the overlapping colored areas for different levels of k. To the right of the lines the combination of α and β produces a larger bias for the semi-parametric approach.

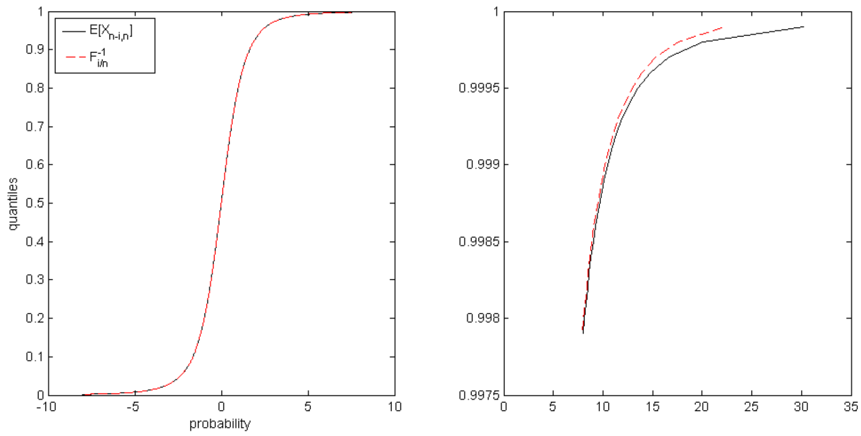
For the parameters of the Symmetric Stable distribution, the bias is always smaller for the semi-parametric estimator, as $\beta = \alpha$ and $\alpha < 2$. In case of the Student-t distribution we have $\alpha \in [1, \infty)$. Therefore, we can find $\alpha > \alpha^*$. From Figure 1, we observe that for $k = 8$ and $\beta = 2$ that the switching of the biases occurs around $\alpha^* = 5$. For higher values of k, the α^* increases as depicted in Figure 1.[10]

[10]For the path of $\log(k)/\sqrt{k}$ see Figure 3 in the Appendix. The maximum of the function is at $k = e^2$.

3 The severity of the problem

To study the severity of the problem under a controlled setting, we use Monte Carlo simulations. We extract samples from parametric distributions that fall in the domain of attraction of the heavy tailed EVT distributions. To analyse the economic implication of the bias, we use the CRSP database to compare the difference between the largest order statistic and the semi-parametric quantile estimator.

Figure 2: Difference between $\mathrm{E}\left[X_{n-k,n}\right]$ and $\mathrm{F}^{-1}\left(\frac{n-k}{n}\right)$



This figure shows the difference between the average order statistic of a sample and the inverse cdf at the appropriate probability level. The sample is drawn from a Student-t (3) distribution of size $n = 10,000$ over $R = 10,000$ repetitions. The left graph depicts the whole support of the empirical distribution. The right graph zooms in on the 20 largest order statistics.

From Figure 2 we see that the wedge between the quantiles of the heavy tailed parametric distribution and the average of the simulated order statistics becomes larger as one moves out into the tail. This bias is in part due to $\Gamma(k - 1/\alpha)$ in the expectation of the order statistics.

The other source for the discrepancy is the discreteness of the empirical distribution. The discreteness of the empirical distribution for the larger order statistics is problematic. At the highest order statistic F_n jumps to 1, while the tail may approach 1 only asymptotically. Hyndman and Fan (1996) argue that the i^{th} order statistic falls in the interval between $\mathrm{F}^{-1}\left(i/n\right)$ and

$\mathrm{F}^{-1}\left({}^{i+1}/_{n}\right)$. They argue in favour of using $p_i = \left(i - \frac{1}{3}\right) / \left(n + \frac{1}{3}\right)$ as the probability to match the i^{th} order statistic. The quantile of a uniform distributed variable is Beta distributed, i.e. $\beta\left(i, n-i+1\right)$. Subsequently, they take the median of the Beta distribution.[11] Therefore, $\mathrm{F}^{-1}\left(1 - \left(i - \frac{1}{3}\right) / \left(n + \frac{1}{3}\right)\right)$, as suggested by Hyndman and Fan (1996), is included as an adjustment for the comparison in the Monte Carlo study.

3.1 Monte Carlo study

For the Monte Carlo simulation we draw from distributions which fall within the domain of attraction of the Fréchet distribution. The specific distributions we use are presented in Table 1.

Table 2 shows that for the heavy tailed distributions the expected maximum is close to the average of the simulated maxima, except for the Symmetric Stable distribution with α close to 1. This is in contrast to $\mathrm{F}^{-1}\left(1 - {}^{1}/_{n}\right)$ which differs considerably from the average of the simulated maxima as reflected by the first two columns of Table 2. Heavy tailed distributions are characterized by having unbounded moments. The large standard deviation of the simulated maximum exemplifies this fact. For the Symmetric Stable distribution the results are less accurate as can be seen from the expectation of the Symmetric Stable distribution. In Table 2 we report the estimates of the semi-parametric quantile estimator from the Monte Carlo study. The semi-parametric quantile estimator has a consistently smaller bias than the non-parametric quantile estimator.[12]

The same conclusions are drawn from the median as a measure of centrality.[13] Due to the positive skewness of the Fréchet distribution, the difference between the empirical median and the quantile function is smaller when compared to the result for the mean. The adjustment suggested by Hyndman and Fan (1996) brings the quantile function close to the theoretical median and empirical median of the maximum. This is expected as the median is transformation invariant.

The tables in the Appendix A.3 report the results for the intermediate order statistics. Tables 8 to 11 report the results for the semi-parametric quantile

[11] They use the one-for-one invariant transformation property of the median as an argument for the median as the centrality measure.

[12] The threshold level k for the semi-parametric estimator is a free parameter. We use the KS distance metric and automated Eye-Ball methodology, further clarified in Danielsson, Ergun, De Haan, and De Vries (2015).

[13] See Table 7 in the Appendix.

Table 2: Simulations maxima for heavy tailed distributions

	True		Non-parametric		Semi-parametric		
	F^{-1}	F^{-1} adj	$\mathrm{E}(X)$	$\mathrm{E}_n(X)$	$F^{-1}_{EVT,KS}$	$\mathrm{F}^{-1}_{EVT,EYE}$	st dev sim
α				Pareto			
2.0	100.0	122.5	177.2	179.9	125.3	103.1	266.8
3.0	21.5	24.7	29.2	29.1	24.5	21.9	18.5
4.0	10.0	11.1	12.3	12.3	11.0	10.1	5.1
5.0	6.3	6.8	7.3	7.4	6.8	6.4	2.5
6.0	4.6	5.0	5.2	5.2	4.9	4.7	1.3
				Student-t			
2.0	70.7	86.6	125.3	124.3	96.5	73.6	174.9
3.0	22.2	25.4	30.1	30.0	27.7	23.5	19.1
4.0	13.0	14.5	16.1	16.1	15.4	14.2	6.9
5.0	9.7	10.5	11.5	11.3	11.1	10.8	3.7
6.0	8.0	8.6	9.4	9.1	9.1	9.2	2.4
				Stable			
1.1	1444.9	2088.9	15179.6	8781.4	2546.4	1635.9	72788.5
1.3	416.8	569.3	1644.2	1632.3	634.3	446.6	17228.8
1.5	158.5	207.7	424.5	404.8	216.7	161.8	1259.6
1.7	68.4	86.7	147.0	148.5	88.1	65.4	400.2
1.9	25.9	32.0	48.1	47.5	31.1	15.1	93.7

This table shows the simulation results for the quantile function, expected maximum and the average of the empirical maximum. The columns with F^{-1} and F^{-1}_{adj} are the inverse quantile function at the probability ${}^{n-1}/_{n}$ and ${}^{((n-1)^{1/3})}/_{(n+1/3)}$, respectively, of the simulated distribution. The column $\mathrm{E}(X)$ gives the analytical expected maximum for the specified distributions. Here $\mathrm{E}_n(X)$ gives the average of the empirical maximum obtained from the simulations. The columns $\mathrm{F}^{-1}_{EVT,KS}$ and $\mathrm{F}^{-1}_{EVT,EYE}$ represent the empirical average of the EVT quantile estimator, where the threshold is determined by the KS distance metric and automated Eye-Ball method, respectively. St dev sim displays the standard deviations of the empirical maximum. The sample size of each simulation is $n = 10,000$ with $R = 10,000$ repetitions.

estimators at ${}^{i}/_{n}$. Tables 13 to 16 report the results for the respective probability levels with the adjustment suggested by Hyndman and Fan (1996). Also for the intermediate order statistics we notice a bias, but the differences are smaller.

To find the point at which the bias for the empirical worst case estimate becomes smaller than the bias for the semi-parametric estimate we simulate from the Student-t distribution with increasing degrees of freedom and a fixed sample size. In Table 4 in the Appendix, we find that the switching point is around $\alpha = 6$ for a sample size of $n = 10,000$. This confirms the theoretical prediction of a shift in the relative size of the bias between the two quantile estimators. This occurs for different sample sizes at different tail index levels, as demonstrated in Tables 5 and 6. The switch of the relative size of the bias is further explored with financial stock market data in the next section.

3.2 Bias in financial returns

In this section we investigate the bias for real world data. We use the CRSP security database to compare the largest order statistic and semi-parametric quantile estimator. This comparison lacks a benchmark as the true quantile is unknown for the security data. Therefore, we focus on the difference between the estimates of the two quantile estimators.

3.2.1 Data

The stock market data is obtained from the Center for Research in Security Prices (CRSP). The CRSP database contains individual stock data from 1925-12-31 to 2013-12-31 for NYSE, AMEX, NASDAQ and NYSE Arca. In the main analysis $n = 2,230$ stocks are used. For every stock that is included in the analysis we require that it is traded on one of the four exchanges during the whole measurement period, which is between 01-01-1995 and 01-01-2011.[14] The fixed time period is to ensure that the sample size is large enough for the EVT estimation.[15] Furthermore, this ensures that the empirical probability at the largest order statistic is the same across the different securities.

3.2.2 Results

For the empirical application the tail exponent α needs to be estimated. To this end we use the Hill estimator. This estimator depends on a selection of a high order statistic as the cut-off for inclusion of extreme observations in the Hill estimator. This nuisance statistic is obtained by the KS distance metric.[16] Given the estimate of the tail index, the quantile can be estimated semi-parametrically. We compare the difference between the non-parametric and semi-parametric estimator at the $1/n$ quantile. The differences are collected in different buckets sorted by $\hat{\alpha}$. This way we are able to determine a switching point in the size of the biases between the two estimators.

[14] In the CRSP database 'exchange code' -2, -1, 0 indicates that a stock was not traded on one of the four exchanges and thus no price data is recorded for these days. Stocks that contain exchange code -2, -1, 0 are not included in the analysis.

[15] The size of the time series for each individual firm is 4030 days.

[16] The KS distance metric is further explained in Danielsson et al. (2015).

Table 3: CRSP data

	All $n = 2,230$	$\hat{\alpha} < 2$ $n = 13$	$2 < \hat{\alpha} < 3$ $n = 712$	$3 < \hat{\alpha} < 4$ $n = 1,001$	$4 < \hat{\alpha} < 5$ $n = 442$	$\hat{\alpha} > 5$ $n = 62$
Mean	-0.003	0.074	0.010	-0.005	-0.017	-0.026
Median	-0.011	0.074	0.005	-0.010	-0.017	-0.026
Ste. dev.	0.031	0.072	0.039	0.022	0.019	0.018
1% quantile	-0.059	-0.038	-0.063	-0.051	-0.063	-0.085
99% quantile	0.109	0.194	0.138	0.064	0.045	0.031
Rank sum test	0.070	0.305	0.397	0.162	0.003	0.047

This table reports the difference between the largest order statistic and semi-parametric quantile estimator for US stocks. These are stocks selected from the CRSP database. The securities need to be traded on NYSE, AMEX, NASDAQ, and NYSE Arca exchanges over the period from 01-01-1995 till 01-01-2011. The table reports various statistics on the distribution of the difference between the two estimators. Here n is the number of different stocks in the different buckets. To determine the number of order statistics for the Hill estimator we use the KS distance metric described in Danielsson et al. (2015).

From the theoretical results and the Monte Carlo simulations, in Table 4 of the Appendix, we see that the relative size of the bias changes as a function of $\hat{\alpha}$. Table 3 reveals a similar pattern when taking the difference of the quantile estimates for the securities in the CRSP database. The switch point is around $\hat{\alpha} = 3$. It is difficult to determine the exact switch point for real data. This is because β, in the bias of the semi-parametric quantile estimator, is not estimated. In addition, the Hill estimator is estimated with a bias. This makes it difficult to determine the switch point. It is encouraging that we see a monotonic decline in the average difference as $\hat{\alpha}$ increases. This is supportive for the result that the bias of the EVT based quantile estimator overtakes the bias of the non-parametric quantile estimator. The results for the median convey the same story.

The 1% and 99% quantiles of the buckets show that although the mean and median showcase a switch between the severity of the bias of the quantile estimators, this might be statistically insignificant. Therefore, we employ the Rank-sum test to test for the difference in size of the observations of the empirical distribution of the EVT quantile estimator and the order statistics. We find that for the lighter heavy tailed random variables the estimates from the estimators are significantly different from one another. The empirical distribution of the semi-parametric quantile estimator tends to have larger values than the distribution of the non-parametric quantile estimator. For low values of $\hat{\alpha}$ the difference is in the expected direction, but insignificant.

4 Conclusion

In this paper we investigate the bias of two different quantile estimators. We concentrate on estimating the maximum of a sample from a heavy tailed distribution. We contrast the order statistics which are the non-parametric quantile estimators and the semi-parametric quantile estimator derived from the scaled Pareto distribution. These estimators are often used in financial applications to calculate the risk of a severe loss. The literature to date on the estimation of the maximum is sparse, and therefore we concentrate on the bias of the maximum.

The biases for both estimators are derived as a function of the tail exponent. This allows us to compare the bias as a function of the heaviness of the tail for a given sample size. We find that for a relatively heavy tail the semi-parametric quantile estimator produces a smaller bias, but as the tail becomes less heavy tailed the largest order statistic is the preferred estimator. We contrast the extent of the bias via Monte Carlo studies and find the predicted switch in the relative bias. US equities data is used to further explore the switching point in the bias. We find that for securities with heavy tails the non-parametric estimator produces on average a larger quantile estimate than the semi-parametric quantile estimator. This switches for the securities with less heavy tails. Although this is not a definite test that this occurs for real world data, it is a strong indication that the predicted effect occurs.

These findings shed new light on the risk measures for market risk currently proposed in the Basel committee consultative report and the stress test in the EIOPA report. They integrate the maximum and the maximum of the ES in their risk measures. When determining the statistical properties of these risk measures the bias of the measurement needs to be taken into account and appropriately dealt with. Furthermore, when doing worst case analysis and estimating the quantile from an empirical sample the same bias is introduced. Given the heaviness of the tail, the appropriate estimator needs to be selected.

References

BIS, 2013. Fundamental review of the trading book: A revised market risk framework. Unpublished working paper. Consultative Document BIS.

Danielsson, J., 2011. Financial risk forecasting: The theory and practice of forecasting market risk with implementation in R and Matlab vol. 587. John Wiley & Sons, .

Danielsson, J., L. Ergun, L. De Haan, C. G. De Vries, 2015. Tail index estimation: Quantile driven threshold selection. Unpublished working paper. Working paper.

Fisher, R. A., L. Tippett, 1928. Limiting forms of the frequency distribution of the largest of smallest member of a sample. in *Proceedings of the Cambridge Philosophical Society* vol. 24 pp. 180–190.

Ghaoui, L. E., M. Oks, F. Oustry, 2003. Worst-case value-at-risk and robust portfolio optimization: A conic programming approach. Operations Research 51(4), 543–556.

Gnedenko, B., 1943. Sur la distribution limite du terme maximum d'une serie aleatoire. Annals of Mathematics 44(1), 423–453.

Goldie, C. M., R. L. Smith, 1987. Slow variation with remainder: Theory and applications. The Quarterly Journal of Mathematics 38(1), 45–71.

Hyndman, R. J., Y. Fan, 1996. Sample quantiles in statistical packages. The American Statistician 50(4), 361–365.

Kerkhof, J., B. Melenberg, H. Schumacher, 2010. Model risk and capital reserves. Journal of Banking & Finance 34(1), 267–279.

Leadbetter, M. R., G. Lindgren, H. Rootzén, 1983. Extremes and related properties of random sequences and processes. Springer Series in Statistics.

Zhu, S., M. Fukushima, 2009. Worst-case conditional value-at-risk with application to robust portfolio management. Operations Research 57(5), 1155–1168.

A Appendix

A.1 $\mathrm{E}(X)$ and median of the Fréchet distribution

We start with the expectation of the distribution with $\alpha > 0$.

$$\mathrm{G}'(x) = g(x) = \alpha a_n^{\alpha} x^{-\alpha-1} e^{-a_n^{\alpha} x^{-\alpha}},$$

for $a_n = n^{\frac{1}{\alpha}}$ we get

$$g(x) = \alpha n x^{-\alpha-1} e^{-n x^{-\alpha}}.$$

To find $\mathrm{E}(X)$ one integrates over the support

$$\mathrm{E}[X] = \int_0^{\infty} x \alpha n x^{-\alpha-1} e^{-n x^{-\alpha}} dx$$

$$= \alpha n \int_0^{\infty} x x^{-\alpha-1} e^{-n x^{-\alpha}} dx.$$

By applying a change of variable $y = n x^{-\alpha}$ and thus $dy = -\alpha n x^{-\alpha-1} dx$

$$\mathrm{E}[X] = \int_{\infty}^{0} -\left(\frac{y}{n}\right)^{-\frac{1}{\alpha}} e^{-y} dy$$

$$= n^{\frac{1}{\alpha}} \int_0^{\infty} \left(\frac{y}{n}\right)^{-\frac{1}{\alpha}} e^{-y} dy.$$

By the Gamma function $\Gamma(q) = \int_0^{\infty} t^{q-1} e^{-t} dt$ we find,

$$\mathrm{E}[X] = n^{\frac{1}{\alpha}} \Gamma\left(1 - \frac{1}{\alpha}\right).$$

Here $\mathrm{E}[X]$ is the expectation you find when one collects the maxima of Pareto distributed samples of size n and average these.

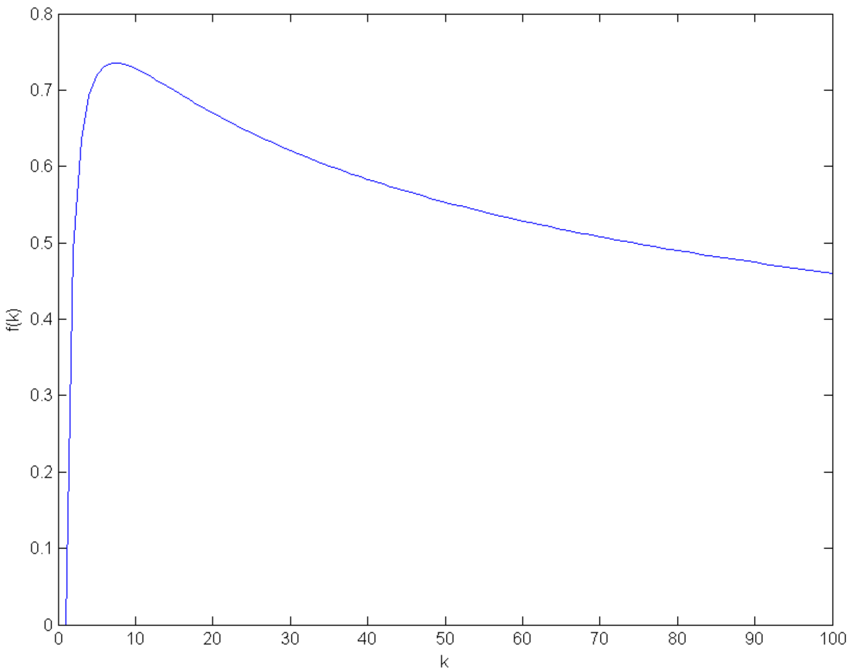
The derivation of the median of the Fréchet distribution is straight forward. We set the cdf equal to a half and solve for x,

$$\mathrm{G}(x) = \exp\left\{-\left(\frac{x - b_n}{a_n}\right)^{-\alpha}\right\} = \frac{1}{2}$$

$$\mathrm{Median}[X] = b_n + a_n \log(2)^{-1/\alpha}.$$

A.2 Figures

Figure 3: $f(k) = \log(k)/\sqrt{k}$



This figure depicts the value of the terms in the bias of the semi-parametric estimator which are dependend on k. The term $\log(k)/\sqrt{k}$ is plotted for various values of k.

A.3 Tables

Table 4: Simulations maxima for Student-t distributions with $n = 10,000$

	True		Non-parametric		Semi-parametric		
α	F^{-1}	F^{-1} adj	$\mathrm{E}(X)$	$\mathrm{E}_n(X)$	Median(X)	Median$_n(X)$	st dev sim
2.000	70.700	86.595	125.331	127.321	96.134	73.119	276.704
3.000	22.204	25.432	30.140	30.064	27.662	23.469	19.299
4.000	13.034	14.450	16.127	16.086	15.488	14.196	6.931
5.000	9.678	10.531	11.521	11.271	11.106	10.813	3.597
6.000	8.025	8.630	9.419	9.115	9.117	9.187	2.445
7.000	7.063	7.535	8.280	7.904	7.985	8.253	1.835
8.000	6.442	6.833	7.600	7.107	7.251	7.648	1.414
9.000	6.010	6.347	7.168	6.576	6.757	7.245	1.241
10.000	5.694	5.993	6.884	6.171	6.395	6.948	1.036
11.000	5.453	5.724	6.694	5.893	6.124	6.718	0.947
12.000	5.263	5.514	6.568	5.661	5.911	6.546	0.854
13.000	5.111	5.344	6.486	5.470	5.740	6.404	0.788
14.000	4.985	5.205	6.435	5.327	5.601	6.301	0.749
15.000	4.880	5.089	6.409	5.193	5.480	6.202	0.701
16.000	4.791	4.991	6.400	5.094	5.384	6.122	0.658
17.000	4.714	4.906	6.405	4.998	5.293	6.053	0.631
18.000	4.648	4.833	6.420	4.917	5.217	6.006	0.605
19.000	4.590	4.769	6.444	4.855	5.153	5.956	0.585
20.000	4.539	4.713	6.475	4.797	5.106	5.897	0.563
21.000	4.493	4.663	6.511	4.738	5.051	5.857	0.548

This table shows the simulation results of the comparison between the quantile function, expected maximum and average of the empirical maximum. The columns with F^{-1} and F^{-1}_{adj} are the inverse quantile function at the probability ${}^{n-1}/{}_{n}$ and ${}^{((n-1)^{1}/3)}/{}_{(n+1/3)}$, respectively, of the simulated distribution. The column $\mathrm{E}(X)$ gives the analytical expected maximum for the Student-t distribution. Here $\mathrm{E}_n(X)$ gives the average of the empirical maximum obtained from the simulations. The columns $\mathrm{F}^{-1}_{EVT,KS}$ and $\mathrm{F}^{-1}_{EVT,EYE}$ represent the empirical average of the EVT quantile estimator, where the threshold is determined by the KS distance metric and automated Eye-Ball method, respectively. St dev sim displays the standard deviations of the empirical maximum. The sample size of each simulation is $n = 10,000$ and with $R = 10,000$ repetitions.

Table 5: Simulations maxima for Student-t distributions with $n = 5,000$

	True		Non-parametric		Semi-parametric		
α	F^{-1}	F^{-1} adj	$\mathrm{E}(X)$	$\mathrm{E}_n(X)$	Median(X)	Median$_n(X)$	st dev sim
2.000	49.985	61.227	88.623	87.458	66.949	52.629	184.209
3.000	17.598	20.164	23.922	24.091	22.027	18.738	18.334
4.000	10.915	12.111	13.561	13.404	12.958	11.903	5.790
5.000	8.363	9.111	10.030	9.790	9.683	9.323	3.147
6.000	7.074	7.618	8.391	8.045	8.081	8.010	2.170
7.000	6.311	6.743	7.500	7.054	7.160	7.251	1.641
8.000	5.811	6.175	6.969	6.438	6.581	6.756	1.375
9.000	5.461	5.778	6.637	5.964	6.159	6.413	1.112
10.000	5.202	5.487	6.423	5.667	5.869	6.148	1.005
11.000	5.004	5.264	6.285	5.415	5.639	5.965	0.922
12.000	4.847	5.089	6.199	5.237	5.476	5.800	0.819
13.000	4.721	4.947	6.149	5.069	5.320	5.683	0.768
14.000	4.616	4.831	6.125	4.940	5.202	5.585	0.717
15.000	4.528	4.733	6.119	4.839	5.101	5.495	0.676
16.000	4.454	4.650	6.129	4.753	5.025	5.421	0.657
17.000	4.390	4.579	6.149	4.668	4.945	5.363	0.619
18.000	4.334	4.517	6.178	4.600	4.884	5.304	0.597
19.000	4.285	4.463	6.213	4.539	4.828	5.265	0.580
20.000	4.241	4.415	6.254	4.492	4.781	5.219	0.552
21.000	4.203	4.372	6.299	4.451	4.738	5.182	0.543

This table shows the simulation results of the comparison between the quantile function, expected maximum and average of the empirical maximum. The columns with F^{-1} and F^{-1}_{adj} are the inverse quantile function at the probability $^{n-1}/_{n}$ and $^{(n-1^{1}/_{3})}/_{(n+^{1}/_{3})}$, respectively, of the simulated distribution. The column $\mathrm{E}(X)$ gives the analytical expected maximum for the Student-t distribution. Here $\mathrm{E}_n(X)$ gives the average of the empirical maximum obtained from the simulations. The columns $\mathrm{F}^{-1}_{EVT,KS}$ and $\mathrm{F}^{-1}_{EVT,EYE}$ represent the empirical average of the EVT quantile estimator, where the threshold is determined by the KS distance metric and automated Eye-Ball method, respectively. St dev sim displays the standard deviations of the empirical maximum. The sample size of each simulation is $n = 5,000$ and with $R = 10,000$ repetitions.

Table 6: Simulations maxima for Student-t distributions with $n = 2,000$

	True		Non-parametric		Semi-parametric		
α	F^{-1}	F^{-1} adj	$\mathrm{E}(X)$	$\mathrm{E}_n(X)$	Median(X)	Median$_n(X)$	st dev sim
2.000	31.599	38.714	56.050	57.136	42.016	33.978	167.008
3.000	12.924	14.820	17.626	17.404	16.018	13.907	10.560
4.000	8.610	9.568	10.785	10.511	10.233	9.384	4.635
5.000	6.869	7.499	8.350	8.103	8.028	7.599	2.898
6.000	5.959	6.434	7.203	6.828	6.879	6.665	1.873
7.000	5.408	5.796	6.580	6.061	6.177	6.103	1.439
8.000	5.041	5.374	6.215	5.582	5.730	5.735	1.188
9.000	4.781	5.076	5.994	5.270	5.435	5.468	1.095
10.000	4.587	4.855	5.861	5.012	5.197	5.264	0.947
11.000	4.437	4.685	5.783	4.825	5.030	5.128	0.856
12.000	4.318	4.550	5.744	4.675	4.888	4.996	0.780
13.000	4.221	4.440	5.730	4.565	4.780	4.905	0.730
14.000	4.140	4.350	5.737	4.451	4.678	4.813	0.706
15.000	4.073	4.273	5.757	4.370	4.603	4.752	0.658
16.000	4.015	4.208	5.787	4.299	4.534	4.699	0.635
17.000	3.965	4.153	5.826	4.231	4.473	4.645	0.606
18.000	3.922	4.104	5.871	4.188	4.429	4.598	0.590
19.000	3.883	4.061	5.921	4.133	4.384	4.564	0.565
20.000	3.850	4.023	5.974	4.099	4.355	4.523	0.553
21.000	3.819	3.989	6.030	4.063	4.315	4.498	0.542

This table shows the simulation results of the comparison between the quantile function, expected maximum and average of the empirical maximum. The columns with F^{-1} and F^{-1}_{adj} are the inverse quantile function at the probability ${}^{n-1}/_{n}$ and ${}^{(n-1^{1}/_{3})}/_{(n+^{1}/_{3})}$, respectively, of the simulated distribution. The column $\mathrm{E}(X)$ gives the analytical expected maximum for the Student-t distribution. Here $\mathrm{E}_n(X)$ gives the average of the empirical maximum obtained from the simulations. The columns $\mathrm{F}^{-1}_{EVT,KS}$ and $\mathrm{F}^{-1}_{EVT,EYE}$ represent the empirical average of the EVT quantile estimator, where the threshold is determined by the KS distance metric and automated Eye-Ball method, respectively. St dev sim displays the standard deviations of the empirical maximum. The sample size of each simulation is $n = 2,000$ and with $R = 10,000$ repetitions.

Table 7: Simulations maxima for heavy tailed distributions

	True		Non-parametric		Semi-parametric		
	F^{-1}	F^{-1} adj	$E(X)$	$E_n(X)$	Median(X)	Median$_n(X)$	st dev sim
α				Pareto			
2.0	100.0	122.5	177.2	179.9	120.1	121.0	266.8
3.0	21.5	24.7	29.2	29.1	24.3	24.4	18.5
4.0	10.0	11.1	12.3	12.3	11.0	11.0	5.1
5.0	6.3	6.8	7.3	7.4	6.8	6.8	2.5
6.0	4.6	5.0	5.2	5.2	4.9	4.9	1.3
				Student-t			
2.0	70.7	86.6	125.3	124.3	84.9	85.3	174.9
3.0	22.2	25.4	30.1	30.0	25.2	25.1	19.1
4.0	13.0	14.5	16.1	16.1	14.4	14.3	6.9
5.0	9.7	10.5	11.5	11.3	10.6	10.5	3.7
6.0	8.0	8.6	9.4	9.1	8.9	8.5	2.4
				Stable			
1.1	1444.9	2088.9	15179.6	8781.4	2016.2	1988.4	72788.5
1.3	416.8	569.3	1644.2	1632.3	552.4	552.1	17228.8
1.5	158.5	207.7	424.5	404.8	202.3	202.0	1259.6
1.7	68.4	86.7	147.0	148.5	84.7	85.9	400.2
1.9	25.9	32.0	48.1	47.5	31.2	31.6	93.7

This table shows the simulation results of the comparison between the quantile function, expected maximum and average of the empirical maximum. The column $E(X)$ gives the analytical expectation of the maximum for the specified heavy tailed distributions. The second column, with $E_n(X)$ gives the average of the empirical maximum obtained from simulation. The columns Median(X) and Median$_n(X)$ represent the theoretical and empirical median for the distance metric, respectively. St dev sim displays the standard deviations of the empirical maximum. The sample size of each simulation is $n = 10,000$ and with $R = 10,000$ repetitions.

Table 8: Simulations $X_{n-2,n}$ for heavy tailed distributions

	True		Non-parametric		Semi-parametric		
	F^{-1}	F^{-1} adj	$E(X)$	$E_n(X)$	$F^{-1}_{EVT,KS}$	$F^{-1}_{EVT,EYE}$	st dev sim
α				Pareto			
2.00	70.71	77.46	88.62	88.78	81.18	71.73	46.00
3.00	17.10	18.17	19.45	19.43	18.50	17.29	5.86
4.00	8.41	8.80	9.19	9.16	8.89	8.46	2.00
5.00	5.49	5.70	5.88	5.88	5.74	5.52	1.03
6.00	4.14	4.26	4.37	4.36	4.28	4.15	0.62
				Student-t			
2.00	49.98	54.76	62.67	63.10	64.11	50.98	32.73
3.00	17.60	18.71	20.09	20.00	21.07	18.26	6.15
4.00	10.92	11.44	12.10	11.96	12.47	11.58	2.73
5.00	8.36	8.69	9.22	8.97	9.29	9.06	1.60
6.00	7.07	7.31	7.85	7.49	7.75	7.81	1.17
				Stable			
1.10	769.53	908.24	1379.96	1384.90	1110.72	801.41	2116.72
1.30	244.59	281.40	379.43	383.74	318.03	250.14	442.85
1.50	99.93	112.83	141.50	140.51	121.12	99.71	117.35
1.70	45.53	50.67	60.54	60.74	52.82	43.82	39.91
1.90	18.06	19.85	22.80	23.00	20.79	12.17	12.40

This table shows the simulation results for the quantile function, expected $X_{n-2,n}$ and average of the empirical order statistic. The columns with F^{-1} and F^{-1}_{adj} are the inverse quantile function at the probability $^{n-2}/_{n}$ and $^{((n-2)^{1/3})}/_{(n+1/3)}$, respectively, of the simulated distribution. The column $E(X)$ gives the analytical expected order statistic for the specified distributions. Here $E_n(X)$ gives the average of the empirical order statistic obtained from the simulations. The columns $F^{-1}_{EVT,KS}$ and $F^{-1}_{EVT,EYE}$ represent the empirical average of the EVT quantile estimator, where the threshold is determined by the KS distance metric and automated Eye-Ball method, respectively. St dev sim displays the standard deviations of the empirical maximum. The sample size of each simulation is $n = 10,000$ with $R = 10,000$ repetitions.

Table 9: Simulations $X_{n-3,n}$ for heavy tailed distributions

	True		Non-parametric		Semi-parametric		
	F^{-1}	F^{-1} adj	$\mathrm{E}(X)$	$\mathrm{E}_n(X)$	$\mathrm{F}^{-1}_{EVT,KS}$	$\mathrm{F}^{-1}_{EVT,EYE}$	st dev sim
α				Pareto			
2.00	57.74	61.24	66.47	66.54	63.81	58.07	24.14
3.00	14.94	15.54	16.21	16.26	15.78	15.05	3.80
4.00	7.60	7.83	8.04	8.04	7.89	7.63	1.31
5.00	5.06	5.19	5.29	5.28	5.22	5.08	0.69
6.00	3.86	3.94	4.00	4.00	3.96	3.88	0.44
				Student-t			
2.00	40.81	43.28	47.00	47.17	50.83	41.18	17.21
3.00	15.35	15.98	16.74	16.64	18.00	15.77	3.78
4.00	9.83	10.14	10.58	10.42	11.02	10.29	1.79
5.00	7.67	7.87	8.30	8.04	8.36	8.17	1.13
6.00	6.56	6.71	7.19	6.81	7.06	7.10	0.81
				Stable			
1.10	532.29	592.47	752.71	750.47	728.25	532.91	616.20
1.30	179.09	196.08	233.49	232.93	221.57	179.07	153.61
1.50	76.30	82.52	94.33	93.57	88.55	75.32	46.10
1.70	35.92	38.48	42.74	43.02	40.00	34.74	19.69
1.90	14.67	15.58	16.80	17.05	16.72	10.75	6.54

This table shows the simulation results for the quantile function, expected $X_{n-3,n}$ and average of the empirical order statistic. The columns with F^{-1} and F^{-1}_{adj} are the inverse quantile function at the probability $^{n-3}/_{n}$ and $^{((n-3)^{1/3})}/_{(n+1/3)}$, respectively, of the simulated distribution. The column $\mathrm{E}(X)$ gives the analytical expected order statistic for the specified distributions. Here $\mathrm{E}_n(X)$ gives the average of the empirical order statistic obtained from the simulations. The columns $\mathrm{F}^{-1}_{EVT,KS}$ and $\mathrm{F}^{-1}_{EVT,EYE}$ represent the empirical average of the EVT quantile estimator, where the threshold is determined by the KS distance metric and automated Eye-Ball method, respectively. St dev sim displays the standard deviations of the empirical maximum. The sample size of each simulation is $n = 10,000$ with $R = 10,000$ repetitions.

Table 10: Simulations $X_{n-4,n}$ for heavy tailed distributions

	True		Non-parametric		Semi-parametric		
	F^{-1}	F^{-1} adj	$E(X)$	$E_n(X)$	$F^{-1}_{EVT,KS}$	$F^{-1}_{EVT,EYE}$	st dev sim
α				Pareto			
2.00	50.00	52.22	55.39	55.47	54.05	50.01	16.52
3.00	13.57	13.97	14.41	14.44	14.12	13.64	2.79
4.00	7.07	7.23	7.37	7.37	7.25	7.10	1.01
5.00	4.78	4.87	4.94	4.93	4.88	4.80	0.54
6.00	3.68	3.74	3.78	3.78	3.74	3.69	0.35
				Student-t			
2.00	35.33	36.91	39.17	39.34	43.23	35.41	11.55
3.00	13.94	14.35	14.88	14.79	16.11	14.21	2.77
4.00	9.13	9.33	9.70	9.52	10.09	9.46	1.37
5.00	7.21	7.35	7.74	7.45	7.76	7.59	0.87
6.00	6.22	6.32	6.80	6.40	6.60	6.63	0.66
				Stable			
1.10	409.82	443.56	524.61	523.13	551.65	400.54	334.54
1.30	143.57	153.50	173.62	173.35	173.98	141.56	85.08
1.50	63.02	66.77	73.37	73.16	71.61	61.79	29.61
1.70	30.37	31.95	34.36	34.61	33.10	29.47	12.41
1.90	12.68	13.25	13.85	14.12	14.41	9.85	4.30

This table shows the simulation results for the quantile function, expected $X_{n-4,n}$ and average of the empirical order statistic. The columns with F^{-1} and F^{-1}_{adj} are the inverse quantile function at the probability ${}^{n-4}/_{n}$ and ${}^{((n-4)^{1/3})}/_{(n+1/3)}$, respectively, of the simulated distribution. The column $E(X)$ gives the analytical expected order statistic for the specified distributions. Here $E_n(X)$ gives the average of the empirical order statistic obtained from the simulations. The columns $F^{-1}_{EVT,KS}$ and $F^{-1}_{EVT,EYE}$ represent the empirical average of the EVT quantile estimator, where the threshold is determined by the KS distance metric and automated Eye-Ball method, respectively. St dev sim displays the standard deviations of the empirical maximum. The sample size of each simulation is $n = 10,000$ with $R = 10,000$ repetitions.

Table 11: Simulations $X_{n-5,n}$ for heavy tailed distributions

	True		Non-parametric		Semi-parametric		
	F^{-1}	F^{-1} adj	$E(X)$	$E_n(X)$	$F^{-1}_{EVT,KS}$	$F^{-1}_{EVT,EYE}$	st dev sim
α				Pareto			
2.00	44.72	46.29	48.47	48.60	47.65	44.55	12.55
3.00	12.60	12.89	13.21	13.25	12.96	12.64	2.23
4.00	6.69	6.80	6.91	6.91	6.80	6.70	0.83
5.00	4.57	4.64	4.69	4.69	4.63	4.59	0.45
6.00	3.55	3.59	3.62	3.62	3.58	3.56	0.29
				Student-t			
2.00	31.60	32.71	34.27	34.35	38.18	31.51	8.76
3.00	12.92	13.23	13.64	13.55	14.78	13.11	2.24
4.00	8.61	8.77	9.10	8.90	9.43	8.87	1.13
5.00	6.87	6.97	7.36	7.06	7.33	7.17	0.74
6.00	5.96	6.04	6.51	6.10	6.27	6.29	0.55
				Stable			
1.10	334.59	356.25	405.38	404.79	449.66	321.68	203.19
1.30	120.95	127.54	140.23	140.44	145.29	118.11	58.84
1.50	54.34	56.89	61.14	61.19	61.04	53.04	21.32
1.70	26.67	27.76	29.30	29.57	28.69	25.96	9.05
1.90	11.33	11.73	12.03	12.33	12.88	9.20	3.19

This table shows the simulation results for the quantile function, expected $X_{n-5,n}$ and average of the empirical order statistic. The columns with F^{-1} and F^{-1}_{adj} are the inverse quantile function at the probability ${}^{n-5}/_{n}$ and ${}^{((n-5)^{1/3})}/_{(n+1/3)}$, respectively, of the simulated distribution. The column $E(X)$ gives the analytical expected order statistic for the specified distributions. Here $E_n(X)$ gives the average of the empirical order statistic obtained from the simulations. The columns $F^{-1}_{EVT,KS}$ and $F^{-1}_{EVT,EYE}$ represent the empirical average of the EVT quantile estimator, where the threshold is determined by the KS distance metric and automated Eye-Ball method, respectively. St dev sim displays the standard deviations of the empirical maximum. The sample size of each simulation is $n = 10,000$ with $R = 10,000$ repetitions.

Table 12: Simulations maxima for heavy tailed distributions (Semi-parametric adjusted)

	True		Non-parametric		Semi-parametric		
	F^{-1}	F^{-1} adj	$E(X)$	$E_n(X)$	$F^{-1}_{EVT,KS}$	$F^{-1}_{EVT,EYE}$	st dev sim
α				Pareto			
2.0	100.0	122.5	177.2	179.9	164.0	127.7	266.8
3.0	21.5	24.7	29.2	29.1	29.0	25.2	18.5
4.0	10.0	11.1	12.3	12.3	12.4	11.2	5.1
5.0	6.3	6.8	7.3	7.4	7.5	6.9	2.5
6.0	4.6	5.0	5.2	5.2	5.3	5.0	1.3
				Student-t			
2.0	70.7	86.6	125.3	124.3	123.9	91.3	174.9
3.0	22.2	25.4	30.1	30.0	32.5	27.2	19.1
4.0	13.0	14.5	16.1	16.1	17.5	16.0	6.9
5.0	9.7	10.5	11.5	11.3	12.4	12.0	3.7
6.0	8.0	8.6	9.4	9.1	10.0	10.1	2.4
				Stable			
1.1	1444.9	2088.9	15179.6	8781.4	4543.7	2508.2	72788.5
1.3	416.8	569.3	1644.2	1632.3	1008.2	630.2	17228.8
1.5	158.5	207.7	424.5	404.8	315.4	215.3	1259.6
1.7	68.4	86.7	147.0	148.5	122.0	82.7	400.2
1.9	25.9	32.0	48.1	47.5	40.3	17.1	93.7

This table shows the simulation results for the quantile function, expected maximum and the average of the empirical maximum. The columns with F^{-1} and F^{-1}_{adj} are the inverse quantile function at the probability $^{n-1}/_{n}$ and $^{((n-1)^{1}/_{3})}/_{(n+^{1}/_{3})}$, respectively, of the simulated distribution. The column $E(X)$ gives the analytical expected maximum for the specified distributions. Here $E_n(X)$ gives the average of the empirical maximum obtained from the simulations. The columns $F^{-1}_{EVT,KS}$ and $F^{-1}_{EVT,EYE}$ represent the empirical average of the EVT quantile estimator, where the threshold is determined by the KS distance metric and automated Eye-Ball method, respectively. The probabilities are adjusted with the suggestion by Hyndman and Fan (1996). St dev sim displays the standard deviations of the empirical maximum. The sample size of each simulation is $n = 10,000$ with $R = 10,000$ repetitions.

Table 13: Simulations $X_{n-2,n}$ for heavy tailed distributions (Semi-parametric adjusted)

	True		Non-parametric		Semi-parametric		
	F^{-1}	F^{-1} adj	$\mathrm{E}(X)$	$\mathrm{E}_n(X)$	$\mathrm{F}^{-1}_{EVT,KS}$	$\mathrm{F}^{-1}_{EVT,EYE}$	st dev sim
α				Pareto			
2.00	70.71	77.46	88.62	88.78	90.73	78.91	46.00
3.00	17.10	18.17	19.45	19.43	19.90	18.40	5.86
4.00	8.41	8.80	9.19	9.16	9.39	8.87	2.00
5.00	5.49	5.70	5.88	5.88	5.99	5.73	1.03
6.00	4.14	4.26	4.37	4.36	4.44	4.28	0.62
				Student-t			
2.00	49.98	54.76	62.67	63.10	71.28	56.13	32.73
3.00	17.60	18.71	20.09	20.00	22.63	19.50	6.15
4.00	10.92	11.44	12.10	11.96	13.19	12.22	2.73
5.00	8.36	8.69	9.22	8.97	9.74	9.49	1.60
6.00	7.07	7.31	7.85	7.49	8.09	8.15	1.17
				Stable			
1.10	769.53	908.24	1379.96	1384.90	1361.18	964.97	2116.72
1.30	244.59	281.40	379.43	383.74	377.59	291.05	442.85
1.50	99.93	112.83	141.50	140.51	140.28	113.18	117.35
1.70	45.53	50.67	60.54	60.74	60.13	48.67	39.91
1.90	18.06	19.85	22.80	23.00	23.02	12.87	12.40

This table shows the simulation results for the quantile function, expected $X_{n-2,n}$ and average of the empirical order statistic. The columns with F^{-1} and F^{-1}_{adj} are the inverse quantile function at the probability ${}^{n-2}/{}_{n}$ and ${}^{((n-2)^{1/3})}/{}_{(n+1/3)}$, respectively, of the simulated distribution. The column $\mathrm{E}(X)$ gives the analytical expected order statistic for the specified distributions. Here $\mathrm{E}_n(X)$ gives the average of the empirical order statistic obtained from the simulations. The columns $\mathrm{F}^{-1}_{EVT,KS}$ and $\mathrm{F}^{-1}_{EVT,EYE}$ represent the empirical average of the EVT quantile estimator, where the threshold is determined by the KS distance metric and automated Eye-Ball method, respectively. The probabilities are adjusted with the suggestion by Hyndman and Fan (1996). St dev sim displays the standard deviations of the empirical maximum. The sample size of each simulation is $n = 10,000$ with $R = 10,000$ repetitions.

Table 14: Simulations $X_{n-3,n}$ for heavy tailed distributions (Semi-parametric adjusted)

	True		Non-parametric		Semi-parametric		
	F^{-1}	F^{-1} adj	$E(X)$	$E_n(X)$	$F^{-1}_{EVT,KS}$	$F^{-1}_{EVT,EYE}$	st dev sim
α				Pareto			
2.00	57.74	61.24	66.47	66.54	68.37	61.74	24.14
3.00	14.94	15.54	16.21	16.26	16.52	15.67	3.80
4.00	7.60	7.83	8.04	8.04	8.16	7.87	1.31
5.00	5.06	5.19	5.29	5.28	5.36	5.21	0.69
6.00	3.86	3.94	4.00	4.00	4.05	3.95	0.44
				Student-t			
2.00	40.81	43.28	47.00	47.17	54.35	43.81	17.21
3.00	15.35	15.98	16.74	16.64	18.84	16.45	3.78
4.00	9.83	10.14	10.58	10.42	11.42	10.65	1.79
5.00	7.67	7.87	8.30	8.04	8.62	8.42	1.13
6.00	6.56	6.71	7.19	6.81	7.25	7.29	0.81
				Stable			
1.10	532.29	592.47	752.71	750.47	820.06	599.57	616.20
1.30	179.09	196.08	233.49	232.93	245.45	197.26	153.61
1.50	76.30	82.52	94.33	93.57	96.82	81.70	46.10
1.70	35.92	38.48	42.74	43.02	43.30	37.16	19.69
1.90	14.67	15.58	16.80	17.05	17.79	11.15	6.54

This table shows the simulation results for the quantile function, expected $X_{n-3,n}$ and average of the empirical order statistic. The columns with F^{-1} and F^{-1}_{adj} are the inverse quantile function at the probability ${}^{n-3}/_{n}$ and ${}^{((n-3)^{1/3})}/_{(n+1/3)}$, respectively, of the simulated distribution. The column $E(X)$ gives the analytical expected order statistic for the specified distributions. Here $E_n(X)$ gives the average of the empirical order statistic obtained from the simulations. The columns $F^{-1}_{EVT,KS}$ and $F^{-1}_{EVT,EYE}$ represent the empirical average of the EVT quantile estimator, where the threshold is determined by the KS distance metric and automated Eye-Ball method, respectively. The probabilities are adjusted with the suggestion by Hyndman and Fan (1996). St dev sim displays the standard deviations of the empirical maximum. The sample size of each simulation is $n = 10,000$ with $R = 10,000$ repetitions.

Table 15: Simulations $X_{n-4,n}$ for heavy tailed distributions (Semi-parametric adjusted)

	True		Non-parametric		Semi-parametric		
	F^{-1}	F^{-1} adj	$E(X)$	$E_n(X)$	$F^{-1}_{EVT,KS}$	$F^{-1}_{EVT,EYE}$	st dev sim
α				Pareto			
2.00	50.00	52.22	55.39	55.47	56.81	52.32	16.52
3.00	13.57	13.97	14.41	14.44	14.60	14.05	2.79
4.00	7.07	7.23	7.37	7.37	7.44	7.25	1.01
5.00	4.78	4.87	4.94	4.93	4.98	4.88	0.54
6.00	3.68	3.74	3.78	3.78	3.80	3.75	0.35
				Student-t			
2.00	35.33	36.91	39.17	39.34	45.39	37.06	11.55
3.00	13.94	14.35	14.88	14.79	16.66	14.67	2.77
4.00	9.13	9.33	9.70	9.52	10.36	9.71	1.37
5.00	7.21	7.35	7.74	7.45	7.94	7.76	0.87
6.00	6.22	6.32	6.80	6.40	6.74	6.77	0.66
				Stable			
1.10	409.82	443.56	524.61	523.13	598.97	436.54	334.54
1.30	143.57	153.50	173.62	173.35	186.97	151.97	85.08
1.50	63.02	66.77	73.37	73.16	76.30	65.60	29.61
1.70	30.37	31.95	34.36	34.61	35.03	30.97	12.41
1.90	12.68	13.25	13.85	14.12	15.06	10.11	4.30

This table shows the simulation results for the quantile function, expected $X_{n-4,n}$ and average of the empirical order statistic. The columns with F^{-1} and F^{-1}_{adj} are the inverse quantile function at the probability $^{n-4}/_{n}$ and $^{((n-4)^{1}/_{3})}/_{(n+^{1}/_{3})}$, respectively, of the simulated distribution. The column $E(X)$ gives the analytical expected order statistic for the specified distributions. Here $E_n(X)$ gives the average of the empirical order statistic obtained from the simulations. The columns $F^{-1}_{EVT,KS}$ and $F^{-1}_{EVT,EYE}$ represent the empirical average of the EVT quantile estimator, where the threshold is determined by the KS distance metric and automated Eye-Ball method, respectively. The probabilities are adjusted with the suggestion by Hyndman and Fan (1996). St dev sim displays the standard deviations of the empirical maximum. The sample size of each simulation is $n = 10,000$ with $R = 10,000$ repetitions.

Table 16: Simulations $X_{n-5,n}$ for heavy tailed distributions (Semi-parametric adjusted)

	True		Non-parametric		Semi-parametric		
	F^{-1}	F^{-1} adj	$E(X)$	$E_n(X)$	$F^{-1}_{EVT,KS}$	$F^{-1}_{EVT,EYE}$	st dev sim
α				Pareto			
2.00	44.72	46.29	48.47	48.60	49.53	46.17	12.55
3.00	12.60	12.89	13.21	13.25	13.31	12.94	2.23
4.00	6.69	6.80	6.91	6.91	6.94	6.82	0.83
5.00	4.57	4.64	4.69	4.69	4.71	4.65	0.45
6.00	3.55	3.59	3.62	3.62	3.63	3.60	0.29
				Student-t			
2.00	31.60	32.71	34.27	34.35	39.67	32.67	8.76
3.00	12.92	13.23	13.64	13.55	15.18	13.44	2.24
4.00	8.61	8.77	9.10	8.90	9.63	9.05	1.13
5.00	6.87	6.97	7.36	7.06	7.46	7.30	0.74
6.00	5.96	6.04	6.51	6.10	6.37	6.39	0.55
				Stable			
1.10	334.59	356.25	405.38	404.79	478.55	344.18	203.19
1.30	120.95	127.54	140.23	140.44	153.52	124.90	58.84
1.50	54.34	56.89	61.14	61.19	64.10	55.60	21.32
1.70	26.67	27.76	29.30	29.57	29.97	27.00	9.05
1.90	11.33	11.73	12.03	12.33	13.33	9.39	3.19

This table shows the simulation results for the quantile function, expected $X_{n-5,n}$ and average of the empirical order statistic. The columns with F^{-1} and F^{-1}_{adj} are the inverse quantile function at the probability ${}^{n-5}/_{n}$ and ${}^{((n-5)^{1/3})}/_{(n+1/3)}$, respectively, of the simulated distribution. The column $E(X)$ gives the analytical expected order statistic for the specified distributions. Here $E_n(X)$ gives the average of the empirical order statistic obtained from the simulations. The columns $F^{-1}_{EVT,KS}$ and $F^{-1}_{EVT,EYE}$ represent the empirical average of the EVT quantile estimator, where the threshold is determined by the KS distance metric and automated Eye-Ball method, respectively. The probabilities are adjusted with the suggestion by Hyndman and Fan (1996). St dev sim displays the standard deviations of the empirical maximum. The sample size of each simulation is $n = 10,000$ with $R = 10,000$ repetitions.